

The University of Chicago

ANALYTICAL INVESTIGATIONS IN
WARING'S THEOREM

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1932

By
RALPH DUNCAN JAMES

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INTRODUCTION

In 1782 E. Waring stated that every integer could be represented as the sum of four squares, nine cubes and in general as a sum of a finite number $g(k)$ of k -th powers. Hilbert in 1909 gave the first complete proof that $g(k)$ exists for every k , but the method gave no indication of the value of $g(k)$. At present only $g(2)$ and $g(3)$ are known.

In a series of papers beginning in 1920, Hardy and Littlewood¹ attacked the problem by means of the modern theory of functions. They introduced a number $G(k)$ defined as the least number such that all integers greater than a certain number $C(k)$ can be represented as the sum of at most $G(k)$ k -th powers. They proved that

$$G(k) \leq (k-2)2^{k-1} + 5,$$

and later that

$$G(k) \leq (k/2 - 1)2^{k-1} + k + 5 + \left[\frac{(k-2)\log 2 - \log k + \log(k-2)}{\log k - \log(k-1)} \right]$$

In 1930 a simpler proof of the first result following the Winogradov method was given by Landau,² in which he generalized

¹ G. H. Hardy and E. J. Littlewood, Some Problems of Partitio Numerorum: I A New Solution of Waring's Problem (Gott. Nach., 1920, pp. 33 - 54), II Proof that every large number is the sum of at most 21 biquadrates (Math. Zeit., 9 (1921), pp. 14 - 27), IV The Singular Series in Waring's Problem and the Value of the Number $G(k)$ (ibid., 12 (1922), pp. 161 - 188), VI Further Researches in Waring's Problem (ibid., 23 (1925), pp. 1 - 37).

² E. Landau, Über die neue Winogradoffsche Behandlungen des Waringschen Problems, Math. Zeit., 31 (1929), pp. 319 - 338.

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the theory to apply to representation by sums of polynomials of degree k . The number $C(k)$ which results from these methods is very large and it is the purpose of this dissertation to modify the original proof so as to reduce $C(k)$.

The proof given here differs from the Landau proof in three main points,

1. In lemma 1 the equality sign holds for one integer m , so that the constant C_3 is actually the best obtainable.

2. In the proof of Theorem 14 (The First Hardy - Littlewood Theorem), a Farey series of order p^{k-2} is employed instead of the usual one of order p^{k-1} , thus considerably reducing the constant C_6 of lemma 2.

3. An approximation due to Van der Corput,¹ which yields a much lower constant, is used in part of the proof instead of the classical result of Weyl.²

Section 1 is concerned with preliminary theorems. The reference after a theorem is to a corresponding theorem in Landau. No proof is given when a theorem is used without change. The first Hardy - Littlewood Theorem is proved in Section 2. Section 3 deals with the singular series and the second Hardy - Littlewood Theorem. References in this section are to Landau's *Vorlesungen über Zahlentheorie*, Bd. 1. The number $C(k)$ is calculated in Section 4 for $k = 4$, $s = 36$; $k = 5$, $s = 57$; $k = 5$, $s = 68$. It is proved that all numbers greater than $10^{26.478}$ are sums of at most 36 fourth powers, all greater than $10^{10^{21.4}}$ are sums of at most 57 fifth powers and all greater than $10^{10^{9.42}}$ are sums of

¹ J. G. Van der Corput, *Zahlentheoretische Abschätzungen*, Math. Zeit., 17 (1923), pp. 250 - 259.

² See Landau, *Vorlesungen über Zahlentheorie*, Bd. 1, Theorem 264, page 253.

at most 68 fifth powers.¹ In Section 5 an assumption, as yet unproved, is made concerning the numbers of solutions of certain diophantine equations. Under this hypothesis it is shown that

$$G(k) \leq 2^k + 1.$$

¹ It is known that for all numbers 37 fourth powers and 58 fifth powers suffice. See, Wiefrich, Über die Darstellung der Zahlen als Summen von Biquadraten, Math. Ann., 66 (1909), pp. 106-108; Zur Darstellung der Zahlen als Summen von 5-ten und 7-ten Potenzen, Math. Ann., 67 (1909), pp. 61-75. Baer, Beiträge zur Waringschen Problem, Dissertation, Göttingen, 1913. Dickson, Generalization of Waring's Theorem for 4-th, 6-th and 8-th powers, Amer. Jour. of Math., 49 (1927), pp. 241-250.

I PRELIMINARY THEOREMS

Notation.

k an integer ≥ 4 ;

$$\sigma = 2^{1-k}; \quad \sigma_1 = 1/(2^{k-2} + 2);$$

$$(1) \quad s, \text{ an integer } \geq \begin{cases} 29, & k = 4, \\ (k-2)2^{k-1} + 5, & k \geq 5; \end{cases}$$

$$\alpha > 0; \quad \beta \geq 0;$$

$$\varphi(x) = \alpha x^k + \alpha_1 x^{k-1} + \dots + \alpha_k$$

an integral valued polynomial in which

$$(2) \quad |\alpha_v| \leq \beta \quad (v = 1, \dots, k);$$

N , an integer > 0 ;

$r(N)$ the number of solutions of

$$(3) \quad \sum_{v=1}^s \varphi(h_v) = N, \quad h_v > 0;$$

$P \geq 1$; z real;

$$S = \sum_{x=1}^{[P]} e^{2\pi i z \varphi(x)};$$

q an integer > 0 ; $(a, q) = 1$;

$$S_{aq} = \sum_{\mu=1}^q e^{2\pi i (a/q) \varphi(\mu)}$$

$$A_q = q^{-s} \sum_{1 \leq a \leq q} S_{aq}^s e^{-2\pi i (a/q) N}; \quad (a, q) = 1$$

$$G_N = \sum_{q=1}^{\infty} A_q;$$

$\delta > 0$, $\theta > 0$, depending only on k and such that

$$\theta + \delta < 1; \quad \theta/2 > \delta \geq \begin{cases} 1/8, & k = 4, \\ 2^{2-k}, & k \geq 5; \end{cases}$$

$c > 0$, depending only on k, s, α, β :

$$J = \int_0^P e^{2\pi i y a u^k} du;$$

$$T = \int_{-P^{-k+\delta}}^{P^{-k+\delta}} e^{-2\pi i c p^k y} y^s dy;$$

$$L = \int_{-\infty}^{\infty} e^{-2\pi i c v} \left(\int_0^1 e^{2\pi i \alpha v w^k} dw \right)^s dv;$$

$$K = \alpha^{-s/k} \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)}.$$

For the definition of symbols not listed above see Landau, Vorlesungen über Zahlentheorie.

THEOREM I. (Landau Theorem 1). The integral L converges and

$$(4) \quad |T - L P^{s-k}| < C_1 P^{s-k-(s/k-1)\delta},$$

$$(5) \quad |T| < C_2 P^{s-k},$$

where

$$(6) \quad C_1 = 2^{s+1} (2/\pi \alpha k)^{s/k} (k/(s-k)),$$

$$(7) \quad C_2 = 2C_1 + (4/\pi \alpha k).$$

LEMMA I.¹ Let $T(m)$ be the number of divisors of m . Then for every $\epsilon_1 > 0$

$$T(m) \leq C_3 m^{\epsilon_1},$$

where

$$C_3 = \frac{2^{\pi(1/\frac{1}{\epsilon_1})} \left(\frac{3}{2}\right)^{\pi(\frac{3}{2})\frac{1}{\epsilon_1}} \left(\frac{4}{3}\right)^{\pi(\frac{4}{3})\frac{1}{\epsilon_1}} \dots}{e^{\epsilon_1 \theta(1/\frac{1}{\epsilon_1})} e^{\epsilon_1 \theta(\frac{3}{2})\frac{1}{\epsilon_1}} e^{\epsilon_1 \theta(\frac{4}{3})\frac{1}{\epsilon_1}} \dots},$$

¹ S. Ramanujan, Proc. London Math. Soc., Series 2, 14 (1915), pp. 347 - 409 (page 392).

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$\pi(x)$ denotes the number of primes not exceeding x , and

$$\theta(x) = \log 2 + \log 3 + \log 5 + \dots + \log p,$$

where p is the greatest prime not exceeding x .

THEOREM II. (Landau Theorem 3).

$$\int_0^1 |s|^4 dz < C_4 P^{2+\epsilon_1},$$

where

$$C_4 = 2k^2(2(\alpha + (k-1)\beta))^{\epsilon_1}, \quad C_3, \\ \epsilon_2 = k\epsilon_1.$$

THEOREM III.¹ Let $m \geq 2$, λ real, z real,

$$\psi(x) = \lambda x^m + \lambda_1 x^{m-1} + \dots + \lambda_m,$$

$$S' = \sum_{x=1}^{[P]} e^{2\pi i z \psi(x)},$$

$$\{\xi\} = \min(\xi - [\xi], [\xi] + 1 - \xi).$$

Then

$$|S'| < 4 \left(P^{1-2^{1-m}} + P^{1-m} 2^{1-m} \left(h_1, \dots, h_{m-1} = 1 \right)^{\min(P, \frac{1}{\{ \lambda z m! h_1 \dots h_{m-1} \}})} \right)^{2^{1-m}}.$$

LEMMA II. Let v be an integer > 0 . Then the number of solutions of

$$h_1 h_2 \dots h_{m-1} = v, \quad 1 \leq h_v \leq P \quad (v = 1, \dots, m-1),$$

is

$$\leq C_6 P^{\epsilon_3},$$

¹ Landau, Vorlesungen über Zahlentheorie, Bd. 1, page 253, Theorem 265.

$$(8) \quad C_6 = C_3^{m-2},$$

and ϵ_3 is given by the recursion formulas

$$(9) \quad \epsilon_3(2) = 0, \quad \epsilon_3(m) = (m-1)\epsilon_1 + \epsilon_3(m/2) + \epsilon_3((m+2)/2),$$

$m \text{ even } \geq 4;$

$$(10) \quad \epsilon_3(3) = 2\epsilon_1, \quad \epsilon_3(m) = (m-1)\epsilon_1 + 2\epsilon_3((m+1)/2), m \text{ odd } \geq 5.$$

Proof 1. Let $m = 2$. Then $h_1 = v$ has at most

$$1 = C_6 P^{\epsilon_3(2)}$$

solutions with $1 \leq h_1 \leq P$.

2. Let $m = 3$. Then

$$h_1 h_2 = v$$

has at most $T(v)$ solutions, since h_1 must divide v . By Lemma 1 we have

$$T(v) \leq C_3 v^{\epsilon_1} \leq C_3 P^{2\epsilon_1} = C_6 P^{\epsilon_3(3)},$$

since

$$v = h_1 h_2 \leq P^2.$$

3. Let m be even, ≥ 4 , and assume the lemma true for all integers $< m$. In the equation

$$(11) \quad h_1 \cdots h_{m-1} = v,$$

write

$$(12) \quad h_1 \cdots h_{(m-2)/2} = v_1,$$

$$(13) \quad h_{m/2} \cdots h_{m-1} = v_2.$$

There are at most $C_3^{m/2-2} P^{\epsilon_3(m/2)}$ solutions of (12) and at most

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$C_3^{(m+2)/2 - 2P\epsilon_3((m+2)/2)}$ solutions of (13). Hence the number of solutions of (11) is

$$\begin{aligned} &\leq T(v)C_3^{m/2 - 2P\epsilon_3(m/2)}C_3^{(m+2)/2 - 2P\epsilon_3((m+2)/2)} \\ &\leq C_3^{P(m-1)\epsilon_3}C_3^{m/2 - 2P\epsilon_3(m/2)}C_3^{(m+2)/2 - 2P\epsilon_3((m+2)/2)} \\ &= C_3^{m-2P(m-1)\epsilon_3 + \epsilon_3(m/2) + \epsilon_3((m+2)/2)} \\ &= C_6^{P\epsilon_3(m)}. \end{aligned}$$

4. Let m be odd, ≥ 5 , and suppose the lemma true for all integers $< m$. As in 3 with

$$\begin{aligned} h_1 \cdots h_{(m-1)/2} &= v_1, \\ h_{(m+1)/2} \cdots h_{m-1} &= v_2, \end{aligned}$$

the number of solutions of (11) is at most

$$\begin{aligned} &T(v)C_3^{(m+1)/2 - 2P\epsilon_3((m+1)/2)}C_3^{(m+2)/2 - 2P\epsilon_3((m+1)/2)} \\ &\leq C_3^{m-2P(m-1)\epsilon_3 + 2\epsilon_3((m+1)/2)} \\ &= C_6^{P\epsilon_3(m)}. \end{aligned}$$

Since the lemma is true for $m = 2$ and $m = 3$ by 1 and 2 respectively the induction is complete.

LEMMA III. Let $z = a/q + y$; $|y| < (1/qP^\gamma)$; $\gamma > 0$; $t > 0$.

Then, for every $\epsilon_4 > 0$,

$$\begin{aligned} &\sum_{h_1, \dots, h_{m-1}=1}^{[P]} \min(P, 1/(tzh_1 \cdots h_{m-1})) \\ (14) \quad &< C_7 P^{\epsilon_4} (P^{m-1} q^{\epsilon_4} + q^{1+\epsilon_4} + tP^{2m-\gamma-1} q^{-1} + \\ &\quad tP^{m-\gamma} + tP^{m-\gamma} q^{-1} + P), \end{aligned}$$

where

$$C_7 = C_6(4/\epsilon_4 e^{1-\epsilon_4}).$$

Proof. Let $(t, q) = \delta$; $t = t_1 \delta$; $q = q_1 \delta$. If $q_1 \leq 2t_1 P^{m-\gamma-1}$, we have

$$\begin{aligned} \sum_{h_1, \dots, h_{m-1}=1}^{[P]} \text{Min} (P, 1/(tzh_1 \dots h_{m-1})) &\leq P^m \leq 2t_1 P^{2m-\gamma-1} q_1^{-1} \\ &= 2t P^{2m-\gamma-1} q^{-1} \\ &< C_7 P^{\epsilon_3} (P^{m-1} q^{\epsilon_4} + q^{1+\epsilon_4} + t P^{2m-\gamma-1} q^{-1} + t P^{m-\gamma} + \\ &\quad t P^{m-\gamma} q^{-1} + P). \end{aligned}$$

Hence we may assume $q_1 > 2t_1 P^{m-\gamma-1}$. By Lemma II,

$$h_1 \dots h_{m-1} = v, \quad 1 \leq h_v \leq P \quad (v = 1, \dots, m-1)$$

has at most $C_6 P^{\epsilon_3}$ solutions. It follows that

$$\begin{aligned} \sum_{h_1, \dots, h_{m-1}=1}^{[P]} \text{Min} (P, 1/(tzh_1 \dots h_{m-1})) \\ (15) \quad &< C_6 P^{\epsilon_3} \sum_{v=1}^{[P]^{m-1}} \text{Min} (P, 1/(t_{zv})) \\ &= C_6 P^{\epsilon_3} \sum_{v=1}^{[P]^{m-1}} \text{Min} (P, 1/(t_1 a_v/q_1 + t_{yv})) . \end{aligned}$$

Divide the sum into q_1 partial sums of at most

$$(P^{m-1}/q_1 + 1)$$

terms each according to the j_1 in

$$t_1 a_v \equiv j_1 \pmod{q_1}, \quad 0 \leq j_1 \leq q_1 - 1.$$

From the definition of $\{\xi\}$ in Theorem III, we have

$$(16) \quad \{t_1 a_v/q_1 + t_{yv}\} = \{j_1/q_1 + t_{yv}\} .$$

1. Let $2t_1 P^{m-\gamma-1} \leq j_1 \leq q_1 - 2t_1 P^{m-\gamma-1}$. Then

$$0 < j_1/2q_1 \leq j_1/q_1 - t_1 P^{m-\gamma-1}/q_1 < j_1/q_1 + t_{yv}$$

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$$< j_1/q_1 + t_1 p^{m-\gamma-1}/q_1 \leq 1/2 + j_1/2q_1 < 1,$$

whence

$$\{j_1/q_1 + tyv\} = \text{Min} (j_1/2q_1, 1 - (1/2 + j_1/2q_1))$$

and

$$(17) \quad \text{Min} (P, 1/(j_1/q_1 + tyv)) \leq \text{Max} (2q_1/j_1, 2q_1/(q_1 - j_1)) .$$

2. Let $0 \leq j_1 < 2t_1 p^{m-\gamma-1}$ or $q_1 - 2t_1 p^{m-\gamma-1} < j_1 \leq q_1 - 1$.

Then

$$(18) \quad \text{Min} (P, 1/(j_1/q_1 + tyv)) \leq P.$$

From (16), (17) and (18), we get

$$\begin{aligned} & \sum_{v=1}^{[P]^{m-1}} \text{Min} (P, 1/(t_1 s v/q_1 + tyv)) \\ & < (p^{m-1}/q_1 + 1) \left(2t_1 p^{m-\gamma-1} \sum_{j_1 \leq q_1 - 2t_1 p^{m-\gamma-1}} \text{Max} (2q_1/j_1, \right. \\ & \quad \left. 2q_1/(q_1 - j_1)) + (4t_1 p^{m-\gamma-1} + 1)P \right) \\ & \leq (p^{m-1}/q_1 + 1) \left[\sum_{1 \leq j_1 \leq q_1/2} 2q_1/j_1 + \sum_{q_1/2 \leq j_1 \leq q_1-1} 2q_1/(q_1 - j_1) \right. \\ & \quad \left. + (4t_1 p^{m-\gamma-1} + 1)P \right] \\ & \leq (p^{m-1}/q_1 + 1) \left[2q_1(1 + \log q_1) + 2q_1(1 + \log q_1) + \right. \\ & \quad \left. 4t_1 p^{m-\gamma} + P \right] \\ & < C_8 (p^{m-1}/q_1 + 1) [q_1^{1+\epsilon_4} + t_1 p^{m-\gamma} + P] \\ & = C_8 (p^{m-1} q_1^{\epsilon_4} + q_1^{1+\epsilon_4} + t_1 p^{2m-\gamma-1} q_1^{-1} + p^m q_1^{-1} + t_1 p^{m-\gamma} + P) \\ & \leq C_8 (p^{m-1} q_1^{\epsilon_4} + q_1^{1+\epsilon_4} + t p^{2m-\gamma-1} q_1^{-1} + t p^m q_1^{-1} + t p^{m-\gamma} + P), \end{aligned}$$

where

$$C_8 = 4/\epsilon_4 e^{1-\epsilon_4}.$$

Hence from (15) it follows that

$$h_1, \dots, h_{m-1} = \sum_{i=1}^{[P]} \text{Min} (P, 1/\{tsh_1 \dots h_{m-1}\}) \\ < C_7 P^{\epsilon_3} (P^{m-1} q^{\epsilon_4} + q^{1+\epsilon_4} + tP^{2m-\gamma-1} q^{-1} + tP^{m} q^{-1} + \\ tP^{m-\gamma} + P),$$

where

$$C_7 = C_6(4/\epsilon_4 e^{1-\epsilon_4}).$$

LEMMA IV.¹ Let $f(x)$ be a real function of x defined on $a \leq x \leq b$, and w a positive integer not exceeding $b - a$. Then

$$(19) \left| \sum_{x=a}^b e^{2\pi i f(x)} \right| < 2((b-a)/w)^{1/2} + ((b-a)/w) \cdot \\ \left(\sum_{r=1}^{w-1} \left| \sum_{x=a}^{b-r} e^{2\pi i (f(x+r)-f(x))} \right| \right)^{1/2}.$$

THEOREM IV. Let

$$S = \sum_{x=1}^{[P]} e^{2\pi i x \varphi(x)},$$

$$x = a/q + y; |y| < 1/qP^{k-\gamma}; \text{ where } \gamma = \begin{cases} 9/5 & k=4, \\ 2 & k \geq 5; \end{cases}$$

$$\sigma = 2^{1-k}.$$

Then

$$(20) |S| < C_9 P^{\epsilon_5} (P^{1+(\gamma-1)\sigma} q^{-\sigma} + P^{1-\sigma} q^{\epsilon_4 \sigma} + P^{1-k\sigma} q^{\sigma+\epsilon_4 \sigma}),$$

¹ E. C. Titchmarsh, On Van der Corput's method and the Zeta Function of Riemann, Quar. Jour. of Math. 2 (1931), pp. 161 - 173, Lemma 3. J. G. Van der Corput, Neue Zahlentheoretische Abschätzungen II, Math. Zeit., 29 (1928 - 1929), pp. 397 - 426, Theorem 1.

where

$$C_9 = 4(1 + (2(\alpha k! + 1)C_7)^\sigma),$$

$$e_5 = \sigma e_3(k).$$

Proof. In Theorem III, let $m = k$, $\lambda = \alpha$, $\lambda_v = \alpha_v$ ($v = 1, \dots, k$). Then $\psi(x) = \phi(x)$ and

$$(21) \quad |S| < 4 \left(P^{1-2^{1-k}} + P^{1-k} 2^{1-k} \left(\sum_{h_1, \dots, h_{k-1}=1}^{[P]} \text{Min}(P, 1/\{\alpha z k! h_1 \dots h_{k-1}\}) \right)^{2^{1-k}} \right).$$

In Lemma III, let $m = k$, $\gamma = k - \delta$, $t = \alpha k!$. Then

$$\begin{aligned} & \sum_{h_1, \dots, h_{k-1}=1}^{[P]} \text{Min}(P, 1/\{\alpha z k! h_1 \dots h_{k-1}\}) \\ & < C_7 P^{e_3} (P^{k-1} q^{e_4} + q^{1+e_4} + \alpha k! P^{k+\delta-1} q^{-1} + \alpha k! P^\delta + \alpha k! P^\delta q^{-1} + P). \end{aligned}$$

Since

$$\alpha k! P^\delta < \alpha k! P^{k-1} q^{e_4},$$

$$P < P^{k-1} q^{e_4},$$

$$\alpha k! P^\delta q^{-1} < \alpha k! P^{k+\delta-1} q^{-1},$$

we have

$$(22) \quad \sum_{h_1, \dots, h_{k-1}=1}^{[P]} \text{Min}(P, 1/\{\alpha z k! h_1 \dots h_{k-1}\}) < C_{10} P^{e_3} (P^{k-1} q^{e_4} + q^{1+e_4} + P^{k+\delta-1} q^{-1}),$$

where

$$C_{10} = 2(\alpha k! + 1)C_7.$$

From (21) and (22), we get

$$|S| < 4(P^{1-\sigma} + P^{1-k\sigma}(C_{10}P^{\epsilon_5} (P^{k-1}q^{\epsilon_4} + q^{1+\epsilon_4} + P^{k-\gamma+1}q^{-1}))^\sigma) \\ < C_9P^{\epsilon_5}(P^{1+(\gamma-1)\sigma}q^{-\sigma} + P^{1-\sigma}q^{\epsilon_4\sigma} + P^{1-k\sigma}q^{\sigma+\epsilon_4\sigma}),$$

where

$$C_9 = 4(1 + C_{10}^\sigma),$$

$$\epsilon_5 = \sigma \epsilon_3(k).$$

THEOREM V. Let $z = a/q + y; |y| < 1/qP^{k-\gamma}$; $\gamma = \begin{cases} 9/5, & k = 4, \\ 2, & k \geq 5; \end{cases}$

$$\sigma = 2^{1-k}$$

$$\sigma_1 = 1/(2^{k-1} + 2)$$

$$|S| = \sum_{x=1}^{[P]} e^{2\pi i z \varphi(x)}.$$

Then

$$(23) |S| < C_{11}P^{\epsilon_6} (P^{1-(2-\gamma)\sigma_1}q^{-\sigma_1} + P^{1-\sigma_1}q^{\epsilon_4\sigma_1} + P^{1-(k-1)\sigma_1}q^{\sigma_1+\epsilon_4\sigma_1}),$$

where

$$C_{11} = 2^{2+\sigma} \left[8(1 + (2(\alpha k_0^1 + 1)C_7)^{2^{2-k}}) \right]^{1/2},$$

$$\epsilon_6 = \sigma \epsilon_3(k-1).$$

Proof. By Lemma IV with $f(x) = \varphi(x)$, $a = 1$, $b = [P]$, we

have

$$(24) |S| < 2 \left(([P] - 1)/w^{1/2} + ([P] - 1)/w \sum_{r=1}^{w-1} \left| \sum_{x=1}^{[P]-r} e^{2\pi i z (\varphi(x+r) - \varphi(x))} \right| \right)^{1/2}.$$

Let

$$\psi(x) = \varphi(x+r) - \varphi(x)$$

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$$\begin{aligned} &= \alpha (x + r)^k + \dots + \alpha_k - \alpha x^k - \dots - \alpha_k \\ &= \alpha k r x^{k-1} + \lambda_1 x^{k-2} + \dots + \lambda_{k-1} \end{aligned}$$

where $\lambda_1, \dots, \lambda_{k-1}$ are polynomials in $\alpha_1, \dots, \alpha_k$ and r .

$$\text{Let } P_1 = [P] - r, \quad S_1 = \sum_{x=1}^{P_1} e^{2\pi i x \psi(x)}.$$

Theorem III applied to S_1 with $m = k - 1$, $\lambda = \alpha k r$, yields

$$\begin{aligned} (25) \quad |S_1| &< 4 \left(P_1^{1-2^{2-k}} + P_1^{1-(k-1)2^{2-k}} \left(\sum_{h_1, \dots, h_{k-2}=1}^{P_1} \text{Min} (P_1, \right. \right. \\ &\quad \left. \left. 1/\{\alpha k! r h_1 \dots h_{k-2}\})^{2^{2-k}} \right) \right). \end{aligned}$$

From Lemma III with $\gamma = k - \delta$, $m = k - 1$, $t = \alpha k! r$, we get

$$\begin{aligned} &\sum_{h_1, \dots, h_{k-2}=1}^{P_1} \text{Min} (P_1, 1/\{\alpha k! r h_1 \dots h_{k-2}\}) \\ &< C_7 P_1^{\epsilon_3} (P_1^{k-2q\epsilon_4} + q^{1+\epsilon_4} + \alpha k! r P_1^{k+\delta-3q-1} + \alpha k! r P_1^{\delta-1} + \\ &\quad \alpha k! r P_1^{\delta-1q-1} + P_1). \end{aligned}$$

Since

$$\alpha k! r P_1^{\delta-1q-1} < \alpha k! r P_1^{k+\delta-3q-1} < \alpha k! r P_1^{k+\delta-3q-1},$$

$$P_1 < P_1^{k-2q\epsilon_4} < P_1^{k-2q\epsilon_4},$$

$$\alpha k! r P_1^{\delta-1} < \alpha k! P P_1^{\delta-1} < \alpha k! P^{\delta} \leq \alpha k! P^{k-2q\epsilon_4},$$

we have

$$\begin{aligned} (26) \quad &\sum_{h_1, \dots, h_{k-2}=1}^{P_1} \text{Min} (P_1, 1/\{\alpha k! r h_1 \dots h_{k-2}\}) \\ &< C_{12} P^{\epsilon_3} (P^{k-2q\epsilon_4} + q^{1+\epsilon_4} + P^{k+\delta-3q-1r}), \end{aligned}$$

where

$$C_{12} = 2(\alpha k! + 1)C_7.$$

Combining (25) and (26) we obtain

$$(27) \quad |S_1| < C_{13} P^{\epsilon_7} (P^{1-2^2-k} \epsilon_4 2^{2-k} + P^{1-(k-1)2^2-k} 2^{2-k} (1 + \epsilon_4) + P^{1+(\gamma-2)2^2-k} 2^{2-k} \epsilon_3 2^{2-k}),$$

where

$$C_{13} = 4(1 + C_{12} 2^{2-k}),$$

$$\epsilon_7 = 2^{2-k} \epsilon_3 (k-1).$$

From (25) and (27) it follows that

$$\begin{aligned} |S| &< 2 \left[P/w^{1/2} + \left((P/w) \sum_{r=1}^{w-1} C_{13} P^{\epsilon_7} (P^{1-2^2-k} \epsilon_4 2^{2-k} + P^{1-(k-1)2^2-k} 2^{2-k} (1 + \epsilon_4) + P^{1+(\gamma-2)2^2-k} 2^{2-k} \epsilon_3 2^{2-k}) \right)^{1/2} \right] \\ &< 2 \left[P/w^{1/2} + (2C_{13})^{1/2} P^{\epsilon_7/2} (P^{1-2^{1-k}} \epsilon_4 2^{1-k} + P^{1-(k-1)2^{1-k}} 2^{1-k} (1 + \epsilon_4) + P^{1+(\gamma-2)2^{1-k}} 2^{1-k} \epsilon_3 2^{1-k}) \right] \\ &< C_{14} P^{\epsilon_6} \left[P/w^{1/2} + P^{1+(\gamma-2)\sigma} q^{-\sigma} w^{\sigma} + P^{1-\sigma} q^{\epsilon_4 \sigma} + P^{1-(k-1)\sigma} q^{\sigma + \epsilon_4 \sigma} \right], \end{aligned}$$

where

$$C_{14} = 2(2C_{13})^{1/2},$$

$$\epsilon_6 = \epsilon_7/2 = \sigma \epsilon_3 (k-1).$$

Choose

$$\begin{aligned} w &= \left[P(2-\gamma)/(2^{k-2}+1) q^{1/(2^{k-2}+1)} \right] + 1 \\ &\leq 2P(2-\gamma)/(2^{k-2}+1) q^{1/(2^{k-2}+1)} < P. \end{aligned}$$

Then

$$|S| < C_{11} p^{\epsilon_1} (p^{1-(2-\gamma)\sigma_1} q^{-\sigma_1} + p^{1-\sigma_1 \epsilon_4 \sigma} + p^{1-(k-1)\sigma_1 \sigma + \epsilon_4 \sigma}),$$

where

$$C_{11} = C_{14} 2^{1+\sigma},$$

$$\sigma_1 = 1/(2^{k-1} + 2).$$

LEMMA V.¹ Let $\alpha = 1/2$, $\beta = 1/2$ be integers where
 $\alpha < \beta$ and let k be an integer ≥ 2 . Let $f(x)$ be a real function of
 x defined on $a \leq x \leq b$, and having derivatives of at least the
first k orders, such that $f^{(k)}(x)$ is always $\geq w$ or $\leq -w$, where
 $w > 0$ and is independent of x . Then

$$\left| \sum_{\alpha < x < \beta} e^{2\pi i f(x)} \right| < C_{15} \left(\int_{\alpha}^{\beta} \sqrt{|f''(u)|} du + 1/\sqrt{w} \right),$$

where

$$C_{15} = \sum_{r=2}^{k-1} 2^{(k-r)(k-r+3)} + (4\sqrt{2} + 2)/(\sqrt{2} - 1).$$

THEOREM VI. Let $z = a/q + y$; $|y| > 0$;

$$S = \sum_{x=1}^{[P]} e^{2\pi i z \varphi(x)}.$$

Then

$$|S| < C_{16} (p^{k/2} q |y| + |y|^{-1/k}),$$

where

$$C_{16} = 2 + ((\alpha k(k-1) + \beta(k-1)(k-2)^2)^{1/2} + (\alpha k!)^{-1/2}) C_{15}.$$

¹ J. G. Van der Corput, Zahlentheoretische Abschätzungen, Math. Zeit., 17 (1923), pp. 250-259, Theorem 4.

Proof.

$$\begin{aligned}
 (28) \quad |S| &= \sum_{x=1}^{[P]} e^{2\pi i x \varphi(x)} = \sum_{\mu=1}^q \sum_{0 \leq \nu \leq [(P-\mu)/q]} e^{2\pi i (a/q + \nu) \varphi(\nu q + \mu)} \\
 &= \sum_{\mu=1}^q e^{2\pi i (a/q) \varphi(\mu)} R(\mu),
 \end{aligned}$$

where

$$(29) \quad R(\mu) = \sum_{0 \leq \nu \leq [(P-\mu)/q]} e^{2\pi i \nu \varphi(\nu q + \mu)}.$$

Also

$$y \varphi(uq + \mu) = y(\alpha(uq + \mu)^k + \dots + \alpha k)$$

$$\begin{aligned}
 |y| |\varphi^k(uq + \mu)| &= |y| |(\alpha k(k-1)(uq + \mu)^{k-2} q^2 + \dots + \\
 &\quad 2\alpha k-2 q^2)|
 \end{aligned}$$

which is by (2)

$$\begin{aligned}
 &< |y| (\alpha k(k-1) + \beta(k-1)(k-2)^2)(uq + \mu)^{k-2} q^2 \\
 &= C_{17} |y| q^2 (uq + \mu)^{k-2};
 \end{aligned}$$

and

$$|y| |\varphi^{(k)}(uq + \mu)| = |y| \alpha k! q^k > 0.$$

Hence by Lemma V with $\alpha = 1/2$, $\beta = [(P - \mu)/q] - 1/2$,

$$f(u) = y \varphi(uq + \mu), \quad w = |y| \alpha k! q^k,$$

we have

$$\begin{aligned}
 &\left| 1/2 \sum_{0 \leq \nu \leq [(P-\mu)/q]} e^{2\pi i \nu \varphi(\nu q + \mu)} \right| \\
 &< C_{15} (C_{17} |y|^{1/2} q \int_0^{(P-\mu)/q} (uq + \mu)^{k/2-1} du + (\alpha k!)^{-1/k} q^{-1} |y|^{-1/k}) \\
 &< C_{15} (C_{17} + (\alpha k!)^{-1/k}) (P^{k/2} |y|^{1/2} + q^{-1} |y|^{-1/k}).
 \end{aligned}$$

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Then

$$\begin{aligned}
 |R(\mu)| &= \left| \sum_{0 \leq \nu \leq 1/2} e^{2\pi i y \varphi(\nu q + \mu)} + \sum_{1/2 < \nu \leq [(P-\mu)/q] - 1/2} e^{2\pi i y \varphi(\nu q + \mu)} \right. \\
 &\quad \left. + \sum_{[(P-\mu)/q] - 1/2 \leq \nu \leq [(P-\mu)/q]} e^{2\pi i y \varphi(\nu q + \mu)} \right| \\
 &< 1 + C_{15}(C_{17} + (\alpha k!)^{-1/k})(P^{k/2}|y|^{1/2} + q^{-1}|y|^{-1/k}) + 1 \\
 &\leq C_{16}(P^{k/2}|y|^{1/2} + q^{-1}|y|^{-1/k}), \text{ independent of } \mu,
 \end{aligned}$$

where

$$C_{16} = 2 + C_{15}(C_{17} + (\alpha k!)^{-1/k}).$$

From (28) we get

$$\begin{aligned}
 |S| &\leq q \max_{1 \leq \mu \leq q} |R(\mu)| \\
 &< C_{16}(P^{k/2}q|y|^{1/2} + |y|^{-1/k}).
 \end{aligned}$$

THEOREM VII. (Landau Theorem 6)

$$|s_{aq}| < C_{18}q^{1-(3-\gamma)\sigma_1 + \epsilon_1},$$

where

$$C_{18} = 3C_{11}.$$

THEOREM VIII. (Landau Theorem 7)

$$q^{-s}|s_{aq}|^s < C_{19}q^{1-(3-\gamma)\sigma_1 s + \epsilon_1 s}$$

where

$$C_{19} = C_{18}^s.$$

THEOREM IX. (Landau Theorem 8)

$$A_q < C_{19}q^{1-(3-\gamma)\sigma_1 s + \epsilon_1 s}.$$

THEOREM X. (Landau Theorem 9)

$$G_N = \sum_{q=1}^{\infty} A_q$$

converges and

$$G_N < C_{20},$$

where

$$C_{20} = C_{19} \sum_{q=1}^{\infty} q^{1-(3-\delta)\sigma_1 s + \epsilon_1 s}.$$

THEOREM XI. (Landau Theorem 10)

$$\left| \sum_{q > P^\theta} A_q \right| < C_{21} P^{-B_1},$$

where

$$C_{21} = ((3 - \delta)\sigma_1 s - \epsilon_1 s - 1) / ((3 - \delta)\sigma_1 s - \epsilon_1 s - 2) C_{19},$$

$$B_1 = \theta((3 - \delta)\sigma_1 s - \epsilon_1 s - 2).$$

THEOREM XII. (Landau Theorem 11) Let Q be an integer > 0 . Then

$$\lim_{Q \rightarrow \infty} \sum_{N=1}^Q G_N s^{s/k-1} = (k/s) Q^{s/k}.$$

THEOREM XIII. (Landau Theorems 12 and 13) Let $z = s/q + y$; $q < P$; $|y| \leq P^{-k+\delta}$. Then

$$\left| s^z - q^{-s} s_{aq}^{s_j s} \right| < C_{22} P^{s-1+\delta+\epsilon_s} q^{1-(3-\delta)\sigma_1(s-1)},$$

where

$$C_{22} = (8\pi(k-1)\beta + 2\pi\alpha k + 3) C_{18}^{s-1},$$

$$\epsilon_s = (s-1)\epsilon_1.$$

LEMMA VI. (Landau Page 337)

$$L = O^{s/k-1}(\alpha^{-s/k} \Gamma^s(1 + 1/k)) / \Gamma(s/k) = O^{s/k-1}_K$$

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THEOREM XIV. For $s \geq \begin{cases} 29, & k = 4, \\ (k-2)2^{k-1} + 5, & k \geq 5, \end{cases}$

$$|r(N) - K \mathfrak{C}_N N^{s/k-1}| < C_{40} N^{s/k-1-B_{17}},$$

where

$$C_{40} = C^{-s/k-1-B_{16}/k} C_{39},$$

$$B_{17} = B_{16}/k.$$

Proof. For $x \geq 1$ we have

$$(30) \quad \varphi(x) \geq \alpha x^k - \beta(x^{k-1} + \dots + 1) \geq \alpha x^k - k\beta x^{k-1} \geq \alpha - k\beta.$$

From (3) we obtain

$$(31) \quad \varphi(h_v) \leq N - (s-1)(\alpha - k\beta) < C_{23}N,$$

where

$$C_{23} = \begin{cases} 1, & \alpha \geq k\beta, \\ 1 - (s-1)(k\beta - \alpha), & \alpha < k\beta. \end{cases}$$

If $h_v < 2k\beta/\alpha$, then $h_v < (2k\beta/\alpha)N^{1/k}$.

If $h_v \geq 2k\beta/\alpha$, then by (30) and (31) we have

$$(\alpha/2)h_v^k < \varphi(h_v) < C_{23}N.$$

$$h_v < ((2/\alpha)C_{23})^{1/k} N^{1/k}.$$

Hence for all $h_v > 0$,

$$h_v < C_{24} N^{1/k},$$

where

$$C_{24} = \text{Max } (2k\beta/\alpha, ((2/\alpha)C_{23})^{1/k}).$$

Let $P = C_{24}N^{1/k} \geq 2$; $N = cP^k$, where $c = C_{24}^{-k}$. Then

$$\begin{aligned} r(N) &= \sum_{1 \leq x_v \leq P} \int_0^1 e^{2\pi i z (\sum_{v=1}^s \varphi(x_v) - N)} dz \\ &= \int_0^1 e^{-2\pi i N z} \prod_{v=1}^s \left(\sum_{0 \leq x_v \leq P} e^{2\pi i z \varphi(x_v)} \right) dz \\ &= \int_0^1 S^s e^{-2\pi i N z} dz. \end{aligned}$$

Divide the interval $(0, 1)$ into subintervals by means of the Farey series

$$a/q, \quad (a, q) = 1, \quad 0 \leq a \leq q \leq P^{k-\delta}.$$

Let $z = a/q + y$, where

$$-\theta_1 \leq y \leq 0 \quad \text{or} \quad 0 \leq y \leq \theta_2,$$

and

$$1/2qP^{k-\delta} \leq \theta_1 \leq 1/qP^{k-\delta}.$$

Consider seven classes of intervals as follows:

Class 1. $1 \leq q \leq P^\theta$, $-P^{-k+\delta} \leq y \leq 0$ or $0 \leq y \leq P^{-k+\delta}$.

Class 2. $P^\theta < q \leq P^{\sigma/\sigma_1 - (2-\delta)}$, $-P^{-k+\delta} \leq y \leq 0$ or $0 \leq y \leq P^{-k+\delta}$.

Class 3. $1 \leq q \leq P^{1/2-1/2k}$, $-P^{-k+1} \leq y \leq -P^{-k+\delta}$ or $P^{-k+\delta} \leq y \leq P^{-k+1}$.

Class 4. $P^{1/2-1/2k} < q \leq P^{\delta-1}$, $-P^{-k+1} \leq y \leq -P^{-k+\delta}$ or $P^{-k+\delta} \leq y \leq P^{-k+1}$.

Class 5. $1 \leq q \leq p^{\delta-1}$, $-\theta_1 \leq y \leq -p^{-k+1}$ or $p^{-k+1} \leq y \leq \theta_2$.

Class 6. $p^{\delta-1} < q \leq p^{\sigma/\sigma_1} \cdot -(2-\delta)$, $-\theta_1 \leq y \leq 0$ or $0 \leq y \leq \theta_2$.

Class 7. $p^{\sigma/\sigma_1} \cdot -(2-\delta) < q \leq p^{k-\delta}$, $-\theta_1 \leq y \leq 0$ or $0 \leq y \leq \theta_2$.

Let

$$H_1 = \sum_1 \int_1 s s_0^{-2\pi i N z} dz,$$

taken over all intervals of the first class. Let

$$H_v = \sum_v \int_v |s|^s dz,$$

taken over all intervals of the v -th class ($2 \leq v \leq 7$). Then

$$(32) \quad |r(N) - H_1| \leq \sum_{v=2}^7 H_v.$$

Class 1. From Theorem XIII we have

$$\begin{aligned} & |s s_0^{-2\pi i N z} - q^{-s} s_{aq} s_0^{-2\pi i (a/q) N_0 - 2\pi i N y_j s}| \\ & < C_{22} p^{s-1+\delta+\epsilon} q^{1-(3-\delta)\sigma_1(s-1)}. \end{aligned}$$

Hence

$$\begin{aligned} & \left| \int_1 s s_0^{-2\pi i N z} - q^{-s} s_{aq} s_0^{-2\pi i (a/q) N} \int_{-p^{-k+\delta}}^{p^{-k+\delta}} e^{-2\pi i N y_j s} dy \right| \\ & < C_{22} p^{s-1+\delta+\epsilon} q^{1-(3-\delta)\sigma_1(s-1)} 2p^{-k+\delta}. \end{aligned}$$

Also $N = op^k$ and

$$\int_{-p^{-k+\delta}}^{p^{-k+\delta}} e^{-2\pi i op^k y_j s} dy = T$$

which is independent of a and q . To each value of q corresponds at most $2q$ values of a , so that

$$\left| H_1 - T \sum_{1 \leq q \leq p^0} \sum_{\substack{1 \leq a \leq q \\ (a, q)=1}} q^{-s} s_{aq} s_0^{-2\pi i (a/q) N} \right|$$

$$\begin{aligned}
&< 2C_{22}p^{s-k-1+2\delta+\epsilon_s} \sum_{1 \leq q \leq p^\theta} 2q \cdot q^{1-(3-\gamma)\sigma_1(s-1)} \\
&\left\{ \begin{aligned} &4C_{22}(4 - (3 - \gamma)\sigma_1(s - 1))/(3 - (3 - \gamma)\sigma_1(s - 1)) \cdot \\ &\quad p^{s-1-k+2\delta+\epsilon_s+\theta(3-(3-\gamma)\sigma_1(s-1))} \quad \text{if } (3 - \gamma)\sigma_1(s - 1) < 3, \\ &4C_{22}(1/\epsilon_q \cdot e^{1-\epsilon_q})p^{s-1-k+2\delta+\epsilon_s+\epsilon_q}, \quad \text{if } (3 - \gamma)\sigma_1(s - 1) = 3, \\ &4C_{22}((3 - \gamma)\sigma_1(s - 1) - 2)/((3 - \gamma)\sigma_1(s - 1) - 3) \cdot \\ &\quad p^{s-1-k+2\delta+\epsilon_s}, \quad \text{if } (3 - \gamma)\sigma_1(s - 1) > 3. \end{aligned} \right. \\
&= C_{23}p^{s-k-B_2},
\end{aligned}$$

where

$$B_2 = \begin{cases} 1 - 2\delta - \theta(3 - (3 - \gamma)\sigma_1(s - 1) - \epsilon_s), & \text{if } (3 - \gamma)\sigma_1(s - 1) < 3, \\ 1 - 2\delta - \epsilon_s - \epsilon_q, & \text{if } (3 - \gamma)\sigma_1(s - 1) = 3, \\ 1 - 2\delta - \epsilon_s, & \text{if } (3 - \gamma)\sigma_1(s - 1) > 3. \end{cases}$$

Since

$$A_q = \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} q^{-s} s_{aq} s_e^{-2\pi i(a/q)N},$$

we have

$$(33) \quad \left| H_1 - T \sum_{q \leq p^\theta} A_q \right| < C_{25} p^{s-k-B_2}.$$

Also

$$(34) \quad \left| T \sum_{q \leq p^\theta} A_q - T \mathcal{G}_N \right| = |T| \left| \sum_{q > p^\theta} A_q \right|$$

which by Theorems I and XI is

$$(35) \quad < C_{26} p^{s-k-B_1},$$

where

$$C_{26} = C_{21} C_2.$$

From Theorems X and I, we obtain

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$$(36) \quad |T \mathfrak{G}_N - L \mathfrak{G}_N P^{s-k}| = |\mathfrak{G}_N| |T - LP^{s-k}| < C_{27} P^{s-k-(s/k-1)\delta},$$

where

$$C_{27} = C_1 C_{20}.$$

Hence (33), (35) and (36) give

$$(37) \quad \begin{aligned} & |H_1 - L \mathfrak{G}_N P^{s-k}| \\ &= |(H_1 - T \sum_{1 \leq q \leq P^0} A_q) + (T \sum_{1 \leq q \leq P^0} A_q - T \mathfrak{G}_N) + (T \mathfrak{G}_N - L \mathfrak{G}_N P^{s-k})| \\ &< C_{28} P^{s-k-B_3}, \end{aligned}$$

where

$$C_{28} = C_{25} + C_{26} + C_{27},$$

$$B_3 = \min (B_1, B_2, (s/k - 1)\delta).$$

Class 2. We have by Theorem V

$$|S| < C_{18} P^{1-(2-\gamma)\sigma_1 + \epsilon_1 q - \sigma_1},$$

$$\int_2 |S|^s dz < C_{18} s P^{s-(2-\gamma)\sigma_1 s + \epsilon_1 s q - \sigma_1 s P^{-k} + \delta}.$$

Hence

$$(38) \quad \begin{aligned} H_2 &= \sum_2 \int_2 |S|^s dz < C_{18} s P^{s-(2-\gamma)\sigma_1 s + \epsilon_1 s - k + \delta} \sum_{P^0 < q \leq P^{\sigma_1/\sigma_1} - (2-\gamma)} 2q \cdot q^{-\sigma_1 s} \\ &< C_{29} P^{s-k-B_4}, \end{aligned}$$

where

$$C_{29} = 2C_{18}^s (\sigma_1 s - 1) / (\sigma_1 s - 2),$$

$$B_4 = (\theta + 2 - \gamma)\sigma_1 s - 2\theta - \delta - \epsilon_1 s.$$

Class 3. From Theorem VI we get

$$|S| < 2C_{16} \max (P^{k/2} q/|y|^{1/2}, |y|^{-1/k})$$

$$\int_3 |S|^s dz < (2C_{16})^s \text{Max} \left(\int_{p-k+\delta}^{p-k+1} p^{sk/2} q^s |y|^{s/2} dy, \int_{p-k+\delta}^{p-k+1} |y|^{-s/k} dy \right) \\ < (2C_{16})^s \text{Max} \left(2/(s+2) p^{s/2-k+1} q^s, \right. \\ \left. (k/(s-k)) p^{s-k+\delta-\delta s/k} \right).$$

Hence

$$H_3 < (2C_{16})^s \text{Max} \left(\sum_{1 \leq q \leq p^{1/2-1/2k}} 2q(2/(s+2)) p^{s/2-k+1} q^s, \right. \\ (39) \quad \left. \sum_{1 \leq q \leq p^{1/2-1/2k}} 2q(k/(s-k)) p^{s-k+\delta-\delta s/k} \right) \\ < C_{30} p^{s-k-B_5},$$

where

$$C_{30} = (2C_{16})^s \text{Max} \left(8/(s+2)^2, 2k/(s-k) \right),$$

$$B_5 = \text{Min} \left((s+2-4k)/2k, (\delta s+1-k(1+\delta))/k \right).$$

Class 4. Let $z = a/q + y$ be any point of an interval of the fourth class. Write

$$z = a_1/q_1 + y_1,$$

where

$$(a_1, q_1) = 1; \quad 1 \leq a_1 \leq q_1 \leq M = 2/q|y| \leq p^{k-\delta-1/2-1/2k}; \\ (40) \quad |y_1| < 1/q_1(2/q|y|) = q|y|/2q_1 \leq q/2q_1 p^{k-1} \leq 1/2q_1 p^{k-\delta}.$$

Then $q_1 > p^{k-2}$. For if

$$p^{\sigma/\sigma, -(2-\gamma)} < q_1 \leq p^{k-2},$$

$$|y_1| \leq 1/2q_1 p^{k-\delta},$$

z would be a point of an interval of the seventh class. Again if

$$1 \leq q \leq p^{\sigma/\sigma, -(2-\gamma)},$$

$$\begin{aligned}
 |aq_1 - a_1q| &= qq_1 |a/q - a_1/q_1| = qq_1 |y - y_1| \leq q_1q|y| + qq_1|y_1| \\
 &< q_1q/p^{k-1} + q^2/2p^{k-1} < p^{\sigma/\sigma}, -(2-\delta)+\delta-1/p^{k-1} + p^{2\delta-2}/2p^{k-1} \\
 &= 1/p^{k-2\delta+2-\sigma/\sigma} + 1/2p^{k-2\delta+1}
 \end{aligned}$$

$$< 1/p + 1/2p < 1, \text{ since } p \geq 2.$$

Since a, q, a_1, q_1 are integers, this implies

$$aq_1 - a_1q = 0.$$

From $(a, q) = 1, (a_1, q_1) = 1$, it follows that

$$a = a_1, \quad q = q_1, \quad y = y_1,$$

whence

$$|y_1| = q|y|/q_1 > q|y|/2q_1,$$

which contradicts (40). Hence Theorem 6 gives

$$\begin{aligned}
 |S| &< 3C_9 p^{\epsilon_s} \text{Max} (p^{1-\sigma} q_1^{\epsilon_4 \sigma}, p^{1-k\sigma} q_1^{\sigma+\epsilon_4 \sigma}) \\
 &\leq 2^{1+\epsilon_4 \sigma} 3C_9 p^{\epsilon_s} \text{Max} (p^{1-\sigma} q^{-\epsilon_4 \sigma} |y|^{-\epsilon_4 \sigma}, p^{1-k\sigma} q^{-(\sigma+\epsilon_4 \sigma)} |y|^{-(\sigma+\epsilon_4 \sigma)}) \\
 \int_4 |S|^s dz &< (2^{1+\epsilon_4 \sigma} 3C_9)^s p^{\epsilon_s s} \text{Max} \left\{ \int_{p^{-k+\delta}}^{p^{-k+1}} p^{s-\sigma s} q^{-\epsilon_4 s} |y|^{-\epsilon_4 s} dy, \right. \\
 &\quad \left. \int_{p^{-k+\delta}}^{p^{-k+1}} p^{s-k\sigma s} q^{-(\sigma s+\epsilon_4 s)} |y|^{-(\sigma s+\epsilon_4 s)} dy, \right. \\
 &< (2^{1+\epsilon_4 \sigma} 3C_9)^s p^{\epsilon_s s} \text{Max} \left\{ \begin{aligned} &(1/(1-\epsilon_4 \sigma s)) p^{s-\sigma s+(k-1)\epsilon_4 \sigma s-k+1} q^{-\epsilon_4 s} \\ &(1/(\sigma s+\epsilon_4 \sigma s-1)) p^{s-k+\delta+(k-\delta)\epsilon_4 \sigma s-\delta \sigma s} q^{-(\sigma s+\epsilon_4 s)} \end{aligned} \right. \\
 H_4 &= \sum_4 \int_4 |S|^s dz \\
 &< (2^{1+\epsilon_4 \sigma} 3C_9)^s p^{\epsilon_s s} \text{Max} \left\{ \begin{aligned} &(1/(1-\epsilon_4 \sigma s)) p^{s-\sigma s+(k-1)\epsilon_4 \sigma s-k+1} \sum_{p^{1/2-1/2k} < q \leq p^{\delta-1}} 2q \cdot q^{-\epsilon_4 s} \\ &(1/(\sigma s+\epsilon_4 \sigma s-1)) p^{s-k+\delta+(k-\delta)\epsilon_4 \sigma s-\delta \sigma s} \sum_{p^{1/2-1/2k} < q \leq p^{\delta-1}} 2q \cdot q^{-(\sigma s+\epsilon_4 s)} \end{aligned} \right.
 \end{aligned}$$

$$(41) \quad < C_{31} p^{s-k-B_6}.$$

where

$$C_{31} = (2^{1+\epsilon_4 \sigma} 30_9)^s \max \left\{ \frac{(3-\epsilon_4 \sigma s)}{(1-\epsilon_4 \sigma s)(2-\epsilon_4 \sigma s)}, \right. \\ \left. 1/(\sigma s + \epsilon_4 \sigma s - 2) \right\},$$

$$B_6 = \min \left\{ \begin{aligned} &\sigma s - 2\gamma + 1 - \epsilon_5 s - (k - \delta) \epsilon_4 \sigma s, \\ &(\delta + 1/2 - 1/2k) \sigma s - (k - \delta - 1/2 + 1/2k) \epsilon_4 \sigma s - \\ &2(1/2 - 1/2k + \delta/2) - \epsilon_5 s. \end{aligned} \right.$$

Class 4. Alternate Proof for $k = 4$, $s \geq 30$, $\delta = 1/8$.

$$(A) \quad p^{51/100} < q \leq p^{4/5}, \quad -p^{-4+1} \leq y \leq -p^{-4+1/8} \text{ or} \\ p^{-4+1/8} \leq y \leq p^{-4+1},$$

$$(B) \quad p^{3/8} < q \leq p^{51/100}, \quad -p^{-4+1} \leq y \leq -p^{-4+31/140} \text{ or} \\ p^{-4+31/140} \leq y \leq p^{-4+1},$$

$$(C) \quad p^{3/8} < q \leq p^{51/100}, \quad -p^{-4+31/140} \leq y \leq -p^{-4+4/25} \text{ or} \\ p^{-4+4/25} \leq y \leq p^{-4+31/140}$$

$$(D) \quad p^{3/8} < q \leq p^{51/100}, \quad -p^{-4+4/25} \leq y \leq -p^{-4+1/8} \text{ or} \\ p^{-4+1/8} \leq y \leq p^{-4+4/25}.$$

As in the proof of (41), we have

$$(42) \quad H_{4A} < C_{31} p^{s-4-B_7}.$$

where

$$B_7 = \min \left\{ \begin{aligned} &s/8 - 18/5 + 1 - (4 - 4/5) \epsilon_4 s/8 - \epsilon_5 s, \\ &(1/8 + 51/100) s/8 + (4 - 1/8 - 51/100) \epsilon_4 s/8 - 2(51/100 + 1/16) \\ &- \epsilon_5 s. \end{aligned} \right.$$

Also

$$\int_{4B} |s|^s dz < (2^{1+\epsilon_4/8} 3C_9)^s P^{\epsilon_5 s} \text{Max} \left\{ \int_{P^{-4+31/140}}^{P^{-4+1}} P^{s-s/8} q^{-\epsilon_4 s/8} |y|^{-\epsilon_4 s/8} dy, \right. \\ \left. \int_{P^{-4+31/140}}^{P^{-4+1}} P^{s-4s/8} q^{-s/8-\epsilon_4 s/8} |y|^{-s/8-\epsilon_4 s/8} dy. \right.$$

Hence

$$(43) \quad H_{4B} < C_{32} P^{s-4-B_8},$$

where

$$C_{32} = 2(2^{\epsilon_4/8} 3C_9)^s (3 - \epsilon_4/8)/(1 - \epsilon_4/8)(2 - \epsilon_4/8),$$

$$B_8 = \text{Min} \left\{ \begin{aligned} &s/8 - 51/50 - 1 - (4 - 151/100)\epsilon_4 s/8 - \epsilon_5 s, \\ &(31/140 + 3/8)s/8 - (4 - 3/140 - 3/8)\epsilon_4 s/8 - 3/4 - 31/140 - \epsilon_5 s. \end{aligned} \right.$$

By the proof of (39)

$$(44) \quad H_{40} < (2C_{16})^s \text{Max} \left\{ \begin{aligned} &P^{3/8} \sum_{q \leq P} 51/100^{2q} (2/(s+2)) P^{31s/140-4+31/140} q^s, \\ &P^{3/8} \sum_{q \leq P} 51/100^{2q} (k/(s-k)) P^{s-4+4/25-s/25} \end{aligned} \right.$$

$$< C_{30} P^{s-4-B_9},$$

where

$$B_9 = \text{Min} \left\{ \begin{aligned} &(1 - 31/280 - 51/100)s - 51/50 - 31/140, \\ &s/25 - 4/25 - 51/50. \end{aligned} \right.$$

By theorem V we get

$$\int_{4D} |s|^s dz < (3C_{11})^s P^{s-s/50+\epsilon_4 s_q-s/10} P^{-4+4/25}.$$

Hence

$$(45) \quad H_{4D} < (3C_{11})^s P^{s-s/50+\epsilon_4 s-4+4/25} \sum_{P^{3/8} < q \leq P} 51/100^{2q} q^{-s/10}$$

$$< C_{35} P^{s-4-B_{10}},$$

where

$$C_{33} = 2(3C_{11})^s (s/10 - 1)/(s/10 - 2),$$

$$B_{10} = (1/5 + 3/8)s/10 - \epsilon_6 s - 4/25 - 3/4.$$

Class 5. As in the proof of (40), let

$$z = a_1/q_1 + y_1, \quad (a_1 q_1) = 1, \quad 1 \leq a_1 \leq q_1 \leq M = 2/q|y|.$$

As before we have $q_1 > p^{k-2}$. Since $q_1 \leq M = 2/q|y| < 2p^{k-1}$, Theorem IV gives

$$\begin{aligned} |S| &< 3C_9 p^{\epsilon_s} 2^{\sigma} p^{1-\sigma} q_1^{\epsilon_4 \sigma} \\ &< 32^{(1+\epsilon_4)\sigma} C_9 p^{1-\sigma+\epsilon_s} q^{-\epsilon_4 \sigma} |y|^{-\epsilon_4 \sigma}. \end{aligned}$$

Hence

$$\begin{aligned} \int_5 |S|^s dz &< (32^{(1+\epsilon_4)\sigma} C_9)^s p^{s-\sigma s+\epsilon_s} s q^{-\epsilon_4 \sigma s} \int_{p^{-k+1}}^{1/q p^{k-\delta}} |y|^{-\epsilon_4 \sigma s} dy \\ &< ((32^{(1+\epsilon_4)\sigma} C_9)^s / (1-\epsilon_4 \sigma s)) p^{s-\sigma s+\epsilon_s \sigma s-k+\delta+(k-\delta)\epsilon_4 \sigma s} q^{-1}, \end{aligned}$$

and

$$H_5 < ((32^{(1+\epsilon_4)\sigma} C_9)^s / (1-\epsilon_4 \sigma s)) p^{s-\sigma s+\epsilon_s \sigma s-k+\delta+(k-\delta)\epsilon_4 \sigma s} \sum_{1 \leq q \leq p^{\delta-1}} 2q \cdot q^{-1}$$

$$(46) \quad < C_{34} p^{s-k-B_{11}},$$

where

$$C_{34} = 2(32^{(1+\epsilon_4)\sigma} C_9)^s / (1 - \epsilon_4 \sigma s),$$

$$B_{11} = \sigma s + 1 - 2\delta - \epsilon_5 s - (k - \delta)\epsilon_4 \sigma s.$$

Class 6. $k = 4$; $k \geq 6$; $k = 5$, $s \geq 55$. By Theorem 7 we have

$$|S| < 3C_{11} p^{1-(2-\delta)\sigma_1 + \epsilon_6 q^{-\sigma_1}},$$

whence

$$\int_5 |S|^s dz < (3C_{11})^s p^{s-(2-\gamma)\sigma, s+\epsilon_1 s-k+\gamma} q^{-\sigma, s-1}$$

and

$$\begin{aligned} H_6 &< (3C_{11})^s p^{s-(2-\gamma)\sigma, s+\epsilon_1 s-k+\gamma} \sum_{p^{1-\gamma} \leq q \leq p^{\sigma/\sigma, -(2-\gamma)}} 2q \cdot q^{-\sigma, s-1} \\ (47) \quad &< C_{35} p^{s-k-B_{12}}, \end{aligned}$$

where

$$C_{35} = 2(3C_{11})^s (\sigma, s/(\sigma, s-1)),$$

$$B_{12} = \sigma, s+1-2\gamma-\epsilon_1 s.$$

Class 6. $k = 5, s \geq 53$

$$(A) \quad p < q \leq p^{9/8}, \quad -p^{-5+7/8} \leq y \leq 0 \text{ or } 0 \leq y \leq p^{-5+7/8},$$

$$(B) \quad p < q \leq p^{9/8}, \quad -\theta_1 \leq y \leq -p^{-5+7/8} \text{ or } p^{-5+7/8} \leq y \leq \theta_2$$

From Theorem V we get

$$\int_{6A} |S|^s dz < (3C_{11})^s p^{s+\epsilon_1 s-5+7/8} q^{-s/18},$$

so that

$$\begin{aligned} H_{6A} &< (3C_{11})^s p^{s+\epsilon_1 s-5+7/8} \sum_{p^{9/8} \leq q \leq p^{9/8}} 2q \cdot q^{-s/18} \\ (48) \quad &< C_{36} p^{s-5-B_{13}}, \end{aligned}$$

where

$$C_{36} = 2(3C_{11})^s (s-18)/(s-36),$$

$$B_{13} = s/18 - 23/8 - \epsilon_1 s.$$

As in the proof of (40) let

$$z = a_1/q_1 + y_1, \quad (a_1, q_1) = 1, \quad 1 \leq a_1 \leq q_1 \leq M = 2/q|y|.$$

Then as before

$$q_1 > p^{5-2}.$$

Also

$$q_1 \leq M \leq 2/q|y| < p^{5-7/8-1} < p^{5-1}.$$

Hence by Theorem IV

$$\begin{aligned} |S| &< 3C_9 p^{1-1/16+\epsilon_s q_1} \epsilon_4/16 \\ &< 2^{\epsilon_4/16} 3C_9 p^{1-1/16+\epsilon_s q_1} \epsilon_4/16 |y|^{-\epsilon_4/16}, \\ \int_{6B} |S|^s dz &< (2^{\epsilon_4/16} 3C_9)^s p^{s-s/16+\epsilon_s s q_1} \epsilon_4 s/16 \int_{p^{-5+7/8}}^{1/q p^{5-2}} |y|^{-\epsilon_4 s/16} dy \\ &< ((2^{\epsilon_4/16} 3C_9)^s / (1 - \epsilon_4 s/16)) p^{s-s/16+\epsilon_s s-5+2+(5-2)\epsilon_4 s/16} \\ &\quad q^{-1}, \end{aligned}$$

and

$$H_{6B} < ((2^{\epsilon_4/16} 3C_9)^s / (1 - \epsilon_4 s/16)) p^{s-s/16+\epsilon_s s-5+2+(5-2)\epsilon_4 s/16} \sum_{p < q \leq p} 9/8 \ 2q \cdot q^{-1}$$

$$(49) \quad < C_{37} p^{s-5-B_{14}},$$

where

$$\begin{aligned} C_{37} &= 2(2^{\epsilon_4/16} 3C_9)^s / (1 - \epsilon_4 s/16), \\ B_{14} &= s/16 - 2 - 9/8 - \epsilon_s s - (5-2)\epsilon_4 s/16. \end{aligned}$$

Class 7. For $q \leq p^{k-2}$, Theorem V gives

$$|S| < 3C_{11} p^{1-\sigma+\epsilon_b q} \epsilon_4 \sigma \leq 3C_{11} p^{1-(\gamma-1)\sigma+\epsilon_b q} \epsilon_4 \sigma.$$

For $p^{k-2} < q \leq p^{k-\gamma}$, we have by the same theorem

$$|S| < 3C_{11} p^{1-(k-1)\sigma + \epsilon_4 q \sigma + \epsilon_4 \sigma} \leq 3C_{11} p^{1-(\gamma-1)\sigma + \epsilon_4 q \epsilon_4 \sigma}.$$

Hence

$$H_7 = \sum_7 \int_7 |S|^s dz \leq \max_7 |S|^{s-4} \sum_7 \int_7 |S|^4 dz \\ < (3C_{11})^{s-4} p^{s-4-(\gamma-1)\sigma(s-4) + \epsilon_4(s-4) + (k-\gamma)\epsilon_4\sigma(s-4)} \int_0^1 |S|^4 dz,$$

which by Theorem III is

$$(50) \quad < (3C_{11})^{s-4} p^{s-4-(\gamma-1)\sigma(s-4) + \epsilon_4(s-4) + (k-\gamma)\epsilon_4\sigma(s-4)} C_4 p^{2+\epsilon_2} \\ < C_{38} p^{s-k-B_{15}},$$

where

$$C_{38} = (3C_{11})^{s-4} C_4, \\ B_{15} = (\gamma-1)\sigma(s-4) - (k-2) - \epsilon_2 - \epsilon_4(s-4) \\ - (k-\gamma)\epsilon_4\sigma(s-4).$$

Combining (32), (37), (38), (39), (41), (46), (47), (50), we get

$$(51) \quad |r(N) - L \odot_N p^{s-k}| < C_{39} p^{s-k-B_{16}},$$

where

$$C_{39} = C_{28} + C_{29} + C_{30} + C_{31} + C_{34} + C_{35} + C_{38} \\ B_{16} = \min(B_3, B_4, B_5, B_6, B_{11}, B_{12}, B_{15}).$$

Since

$$N = Cp^k, \quad p = C^{-1/k} N^{1/k},$$

this becomes

$$(52) \quad |r(N) - C^{-(s/k-1)} L \odot_N N^{s/k-1}| < C^{-s/k-1-B_{16}/k} C_{39} N^{s/k-1-B_{16}/k}$$

By Lemma VI

$$C^{-(s/k-1)} L = K.$$

Hence

$$(53) \quad |r(N) - K \zeta_N^{s/k-1}| < c_{40} N^{s/k-1-B_{17}},$$

where

$$c_{40} = c^{-s/k-1-B_{16}/k} c_{39}.$$

$$B_{17} = B_{16}/k.$$

3. THE SINGULAR SERIES. THE SECOND HARDY LITTLEWOOD THEOREM

Notation.

$$p \text{ a prime; } X_p(N) = \sum_{\gamma=0}^{\infty} A_p \gamma;$$

Θ , the highest power of p which divides k ;

$$\gamma = \begin{cases} \Theta + 2, & p = 2, \\ \Theta + 1, & p > 2; \end{cases}$$

$$P = p^\gamma; \quad r = ((P - 1)/(p - 1))k, \quad p - 1);$$

$N(p^\gamma, N)$, the number of solutions of

$$\sum_{v=1}^s h_v^k \equiv N \pmod{p^\gamma}, \quad 0 \leq h_v < p^\gamma,$$

not every h_v divisible by p .

THEOREM XV. (Landau Theorem 301 (2)). For $s \geq r + 1$ and
for every N .

$$N(P, N) > 0.$$

THEOREM XVI. (Landau, proof of Theorem 302)

$$X_p(N) \geq P^{1-s} \min(N(P, N), N(P, 0)).$$

THEOREM XVII. For every N ,

$$X_p(N) > P^{-r} = b(p).$$

Proof.

$$h_1^k + \dots + h_{r+1}^k \equiv N - h_{r+2}^k - \dots - h_s^k \pmod{P}$$

has, by Theorem XV, at least one solution $h_1, \dots, h_{r+1}$ for every set of values of $h_{r+2}, \dots, h_s$. Hence for every N

$$N(P, N) \geq P^{s-r-1}.$$

With Theorem XVI this gives

$$X_p(N) \geq P^{1-s} P^{s-r-1} = P^{-r}.$$

THEOREM XVIII. (Landau Theorem 319). For $s \geq 4$, $\mathfrak{G}_N$ converges absolutely, and

$$\mathfrak{G}_N = \prod_p X_p(N).$$

THEOREM XIX. (Landau Theorem 324). For $s \geq 5$,
 $p > (1 + ks)^{2/(s-6)},$

$$X_p(N) > 1 - p^{-s}.$$

THEOREM XX. (Landau Theorem 325). For $s \geq (k-2)2^{k-1} + 5$,

$$\mathfrak{G}_N > b_4.$$

where

$$b_4 = \prod_{p < (1+ks)^{2/(s-6)}} b(p) \prod_{p > (1+ks)^{2/(s-6)}} (1 - p^{-2}).$$

THEOREM XXI. For $s \geq \begin{cases} 29 & k = 4, \\ (k-2)2^{k-1} + 5, & k \geq 5. \end{cases}$ $N > C(k),$

$$r(N) > 0.$$

Proof. From Theorems XIV and XX, it follows that

$$\begin{aligned} r(N) &> K \mathfrak{G}_N N^{s/k-1} - C_{40} N^{s/k-1-B_{17}} \\ &> Kb_4 N^{s/k-1} - C_{40} N^{s/k-1-B_{17}} \\ &> 0, \end{aligned}$$

provided

$$N > (C_{40}/Kb_4)^{1/B_{17}} = C(k).$$

4. DETERMINATION OF THE NUMBER $C(k)$

The only computation of any difficulty is the evaluation of

$$C_3 = \frac{2^{\pi(2^{1/\epsilon_1})} \cdot \left(\frac{3}{2}\right)^{\pi(\frac{3}{2})^{1/\epsilon_1}} \cdots}{e^{\epsilon_1 \theta(2^{1/\epsilon_1})} e^{\epsilon_1 \theta(\frac{3}{2})^{1/\epsilon_1}} \cdots}$$

If the numbers $n^{1/\epsilon}$, ($n = 2, 3/2, 4/3, \dots$) fall within Lehmer's table,¹ $\pi n^{1/\epsilon}$, and $\theta(n^{1/\epsilon})$ may be readily found. In case $n^{1/\epsilon}$, is not within the table, we employ the approximations²

$$\pi(n^{1/\epsilon}) < (121/100) \Delta(n^{1/\epsilon}, / \log n^{1/\epsilon}),$$

$$\theta(n^{1/\epsilon}) > \Delta n^{1/\epsilon} - (12/5) \Delta n^{1/2\epsilon} - (5/8 \log 6) \log^2 n^{1/\epsilon} - (15/4) \log n^{1/\epsilon} - 3,$$

where

$$\Delta = \log 2^{1/2} 3^{1/3} 5^{1/5} / 30^{1/30}.$$

The results for three cases are given in the following table:

	k	s	σ	σ_1	$\varphi(x)$	δ	θ	c	γ	ϵ_1	ϵ_2	$C(k)$
1.	4	36	1/8	1/10	x^k	1/8	27/32	1	9/5	1/15	1/1000	$10^{26.478}$
2.	5	57	1/16	1/18	x^k	1/8	27/32	1	2	1/71	1/1000	$10^{10^{21.4}}$
3.	5	68	1/16	1/18	x^k	1/10	22/25	1	2	1/30	1/1000	$10^{10^{9.24}}$

¹ D. N. Lehmer, List of Prime Numbers from 1 to 10,006,721, Carnegie Institute of Washington Publication No. 165, 1914.

² E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Bd. 1, page 24. D. N. Lehmer, List of Prime Numbers, Introduction, page VII.

Hence every number $> 10^{26.478}$ can be represented as the sum of at most 36 fourth powers; every number $> 10^{10^{21.4}}$, as the sum of at most 57 fifth powers; and every number $> 10^{10^{9.24}}$, as the sum of at most 68 fifth powers.¹

¹ It is known that all numbers require not more than 37 fourth powers or 58 fifth powers.

5. A FURTHER REMARK

In the proof of the first Hardy Littlewood Theorem given by Landau,¹ there is just one point at which the restriction $s \geq (k-2)2^{k-1} + 5$ is imposed and that is in considering intervals of Class II.² At all other places $s \geq 2^k + 1$ is sufficient.

Hardy and Littlewood³ made the hypothesis:

The number of representations of N by k k-th powers is
 $< cN^\epsilon$ for every $\epsilon > 0$.

With this assumption the restriction is easily removed. In fact Hardy and Littlewood have proved considerably more. By making a somewhat different and probably weaker hypothesis, we shall arrive at the result that it suffices to take $s \geq 2^k + 1$.

Let $R(k, P, s)$ denote the number of solutions of

$$(54) \quad x_1^k + \dots + x_s^k = u_1^k + \dots + u_s^k, \quad \begin{cases} 0 \leq x_1 \leq P, \\ 0 \leq u_1 \leq P. \end{cases}$$

Note that

$$\begin{aligned} \int_0^1 |S|^{2s} dz &= \int_0^1 \left| \sum_{x=1}^P e^{2\pi i z x^k} \right|^{2s} dz \\ &= \sum_{\substack{u, x, \\ 1 \leq v \leq s}} \int_0^1 e^{2\pi i z (x_1^k + \dots + x_s^k - u_1^k - \dots - u_s^k)} dz \end{aligned}$$

¹ E. Landau, Über die neue Winogradoffsche Behandlung des Waringschen Problems, Math. Zeit. (31) 1929, pp. 319-338. We consider here the special case $\varphi(x) = x^k$.

² This refers to Landau's division, not to the one used in Theorem XIV.

³ G. H. Hardy and E. J. Littlewood, Some Problems of Partitio Numerorum; VI. Further Researches in Waring's Problem, Math. Zeit., 23 (1925), pp. 1-37.

$$= R(k, P, s).$$

We make the assumption that

$$(55) \quad R(k, P, 2^{k-2}) \leq R(k-1, P, 2^{k-2})$$

for every $k \geq 2$.

LEMMA VI. Let

$$S_k = \sum_{x=1}^{[P]} e^{2\pi i s x^k};$$

$$s = a/q + y; \quad (a, q) = 1; \quad 1 \leq a \leq q \leq P^{k-1};$$

1. $1 \leq q \leq P, \quad |y| \leq P^{-k},$
2. $1 \leq q \leq P, \quad |y| > P^{-k},$
3. $P < q \leq P^{k-1}.$

Then

$$(56) \quad \sum_{1,2} \int |S_k|^{2k} ds < E_{1,k} P^{2k-k+\epsilon_{1,k}},$$

where

$$E_{1,k} = 2(D_{2,k})^{2k} \text{ of Landau Theorem 5) } C_8,$$

$$\epsilon_{1,k} = 2^k (\epsilon \text{ of Landau Theorem 5) } + \epsilon_4.$$

Also

$$(57) \quad \int_0^1 |S_k|^{2k} ds < E_k P^{2k-k+\epsilon_k},$$

where

$$E_2 = C_4 \text{ of Theorem II,}$$

$$E_k = E_{1,k} + D_{2,k} 2^{k-1} E_{k-1}, \quad k \geq 3;$$

$$\epsilon_2 = \epsilon_1 \text{ of Theorem II,}$$

$$\epsilon_k = \max (\epsilon_{1,k} , 2^{k-1} \epsilon + \epsilon_{k-1}), \quad k \geq 3.$$

Proof. 1. $1 \leq q \leq P$, $|y| < P^{-k}$. Then Landau Theorem 5 gives

$$\int_1 |s_k|^{2^k} dz < D_{2,k} 2^k P^{2^k + 2^k \epsilon} q^{-2P-k}$$

$$\sum_1 \int |s_k|^{2^k} dz < D_{2,k} 2^k C_8 P^{2^k - k + 2^k \epsilon + \epsilon_4},$$

where C_8, ϵ_4 are the same as in Lemma III.

2. $1 \leq q \leq P$, $|y| > P^{-k}$. As in the proof of Theorem XIV, Class 4, we write

$$z = a_1/q_1 + y_1; \quad (a_1, q_1) = 1; \quad 1 \leq a_1 \leq q_1 \leq M = 2/q|y|.$$

Then

$$\int_2 |s_k|^{2^k} dz < D_{2,k} 2^k P^{2^k - 2k + 2^k \epsilon} q^{-2} \int_{P^{-k}}^{1/qP^{k-1}} |y|^{-2} dy$$

$$= D_{2,k} 2^k P^{2^k - 2k + 2^k \epsilon + k} q^{-2}.$$

Hence

$$\sum_2 \int |s_k|^{2^k} dz < D_{2,k} 2^k C_8 P^{2^k - k + 2^k \epsilon + \epsilon_4}$$

and

$$\sum_{1,2} \int |s_k|^{2^k} dz < 2D_{2,k} 2^k C_8 P^{2^k - k + 2^k \epsilon + \epsilon_4}.$$

This proves (56).

Let $k = 2$. Then by Theorem II

$$\int_0^1 |s_2|^4 < C_4 P^{2 + \epsilon_1}.$$

Let $k \geq 3$ and assume (57) true for $k - 1$.

3. $P < q \leq P^{k-1}$. By Landau Theorem 5 we have

$$\begin{aligned}
\sum_3 \int |s_k|^{2^k} dz &< D_{2,k} 2^{k-1} p^{2^{k-1}-1+2^{k-1}\epsilon} \sum_3 \int |s_k|^{2^{k-1}} dz \\
&\leq D_{2,k} 2^{k-1} p^{2^{k-1}-1+2^{k-1}\epsilon} \int_0^1 |s_k|^{2^{k-1}} dz \\
&= D_{2,k} 2^{k-1} p^{2^{k-1}-1+2^{k-1}\epsilon} R(k, p, 2^{k-2}) \\
&\leq D_{2,k} 2^{k-1} p^{2^{k-1}-1+2^{k-1}\epsilon} R(k-1, p, 2^{k-2})
\end{aligned}$$

(by our assumption (55))

$$\begin{aligned}
&= D_{2,k} 2^{k-1} p^{2^{k-1}-1+2^{k-1}\epsilon} \int_0^1 |s_{k-1}|^{2^{k-1}} dz \\
&< D_{2,k} 2^{k-1} p^{2^{k-1}-1+2^{k-1}\epsilon} E_{k-1} p^{2^{k-1}-(k-1)+\epsilon_{k-1}} \\
&= D_{2,k} 2^{k-1} E_{k-1} p^{2^{k-1}-k+2^{k-1}\epsilon+\epsilon_{k-1}}.
\end{aligned}$$

Combining this result with (56) we obtain

$$\int_0^1 |s|^{2^k} dz < E_k p^{2^k-k+\epsilon_k},$$

where

$$E_k = E_{1,k} + D_{2,k} 2^{k-1} E_{k-1},$$

$$\epsilon_k = \max(\epsilon_{1,k}, k, 2^{k-1}\epsilon + \epsilon_{k-1}).$$

THEOREM XXII. For $s \geq 2^k + 1$,

$$|r(N) - K \zeta_N^{s/k-1}| < CN^{s/k-1-B}.$$

Proof. In view of the remark at the beginning of this section, it suffices to show that

$$\sum_{II} \int_{II} |s|^s dz < CP^{s-k-kB},$$

for $s \geq 2^k + 1$. We have

$$\sum_{II} \int_{II} |s|^s dz < D_{2,k} s^{-2^k} p^{s-2^k+(s-2^k)\epsilon-\sigma(s-2^k)} \int_0^1 |s|^{2^k} dz,$$

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which by Lemma VII is

$$\begin{aligned} & \langle D_{2,k}^{s-2k} P^{s-2k+(s-2k)c-\sigma(s-2k)} E_k P^{2k-k+c_k} \\ & \langle CP^{s-k-kB}, \end{aligned}$$

where

$$\begin{aligned} C &= D_{2,k}^{s-2k} E_k, \\ kB &= \sigma(s-2k) - c_k - (s-2k)c. \end{aligned}$$

VITA

Ralph Duncan James was born near Liverpool, England on February 8, 1909. He received his elementary and high school education in the Liverpool schools and at the Prince of Wales High School in Vancouver, Canada. In 1924 he entered the University of British Columbia at Vancouver, graduating in May, 1928 with the degree of Bachelor of Arts. For the next two years he was Assistant in Mathematics and Physics at the same institution and was granted the degree of Master of Arts in May, 1930. He was in residence at the University of Chicago as Cyrus S. Eaton Fellow in Mathematics and as a Fellow in Mathematics for seven consecutive quarters beginning with the Autumn Quarter, 1930. At the University of British Columbia he studied under Professors Buchanan, Nowlan, and Richardson, and at the University of Chicago under Professors Albert, Barnard, Bartky, Bliss, Dickson, Graves, Hille, Lane, Logsdon and Mac Millan. He wishes to express his indebtedness to all his instructors and especially to Professor L. E. Dickson under whose direction this thesis was written.

